ELSEVIER Journal of Geometry and Physics 28 (1998) 289–319

JOURNAL OF GEOMETRY AND PHYSICS

Divisors on elliptic Calabi–Yau 4-folds and the superpotential in F-theory – I ★

A. Grassi

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA

Received 19 June 1997

Abstract

Each smooth elliptic Calabi–Yau 4-fold determines both a three-dimensional physical theory (a compactification of "M-theory") and a four-dimensional physical theory (using the "F-theory" construction). A key issue in both theories is the calculation of the "superpotential" of the theory, which by a result of Witten is determined by the divisors D on the 4-fold satisfying $\chi(\mathcal{O}_D) = 1$. We propose a systematic approach to identify these divisors, and derive some criteria to determine whether a given divisor indeed contributes. We then apply our techniques in explicit examples, in particular, when the base B of the elliptic fibration is a toric variety or a Fano 3-fold.

When B is Fano, we show how divisors contributing to the superpotential are always "exceptional" (in some sense) for the Calabi–Yau 4-fold X. This naturally leads to certain transitions of X, i.e., birational transformations to a singular model (where the image of D no longer contributes) as well as certain smoothings of the singular model. The singularities which occur are "canonical", the same type of singularities of a (singular) Weierstrass model. We work out the transitions. If a smoothing exists, then the Hodge numbers change.

We speculate that divisors contributing to the superpotential are always "exceptional" (in some sense) for X, also in M-theory. In fact we show that this is a consequence of the (log)-minimal model algorithm in dimension 4, which is still conjectural in its generality, but it has been worked out in various cases, among which are toric varieties. © 1998 Elsevier Science B.V.

Subj. Class.: Strings
1991 MSC: 14 99; 14D99
Keywords: String theory; F-theory; Superpotential; Birational contractions

★ Research supported in part by NSF grant DMS-94-01495.

PII S0393-0440(98)00004-7

0. Outline of the paper

The general framework for this paper is given by the string theories and the dualities among them (see for example [10,40,42]. The original motivation and various applications of this work come from physics, while the techniques used are in the realm of algebraic geometry.

We rely in fact on the work of Witten, who in [44] gives necessary and sufficient conditions, in mathematical terms, for the objects of this research, the divisors contributing to what is known as *the non-perturbative superpotential in F-theory* (see also [5,6,8,9,12,16, 17,21,28,29,45]).

We propose a systematic approach to identify these (smooth) irreducible divisors and show how this leads to questions in (birational) algebraic geometry.

F-theory, introduced by Vafa [41], expoits the non-perturbative $SL(2,\mathbb{Z})$ symmetry of type IIB string theory in order to produce new types of physical models associated with elliptic fibrations. These F-theory models can be regarded as string theories which have been "compactified" on varieties which admit an elliptic fibration, often assumed to have a section; the $SL(2,\mathbb{Z})$ is identified (under the duality between F- and IIB-theory) with the symmetry of the homology of the generic fiber. Our results are phrased in the context of F-theory (nevertheless, many of the properties stated here are also true in a related theory known as M-theory).

We thus consider a smooth elliptic Calabi–Yau 4-fold $\pi : X \to B$ with a section, without loss of generality we can assume that $\mu : X \to W$ is the resolution of a Weierstrass model $\pi_0 : W \to B$ (cf. Lemma 1.2) and that B is uniruled.

Each smooth elliptic Calabi-Yau 4-fold determines both a three-dimensional physical theory (a compactification of "M-theory") and a four-dimensional physical theory (using the "F-theory" construction). A key issue in both theories is the calculation of the "*superpotential*" of the theory, a sum

$$S(z) = \sum_{D} \exp\langle c(D), z\rangle,$$

over certain smooth complex divisors $D \subset X$, where X is a smooth Calabi–Yau 4-fold and $z \in H_2(X,\mathbb{Z})$; a necessary condition for D to contribute to the superpotential is $\chi(D) = \chi(D, \mathcal{O}_D) = 1$ [44].

In F-theory, a divisor D contribues to the superpotential only if it is "vertical", i.e., $\pi(D)$ is a proper subset of B; if the fibration is equidimensional, then such divisors are either components of the singular fibers (in this case W is necessarily singular), or of the form $D = \pi^*(C)$, for some smooth divisor C on the 3-fold B (Section 1). We observe that the divisors of the first type are "*exceptional*" for μ (in a sense defined precisely in Observation 1.3 and Section 6), are always *finite* in number, and can be analyzed by using "ad hoc" methods, starting with Kodaira's analysis of the singular fibers and exceptional divisors of Calabi–Yau 4-folds. This is the approach of Katz and Vafa [17]. We study here divisors of the second type.

In particular we focus on two questions; namely determining when the number of such divisors is finite and when D is the exceptional divisor of a birational morphism, which seems to be the case in most examples [5,21,28,29,44].

We show how these questions, which are of interest in physics, naturally lead to other (open) questions in birational algebraic geometry. For example, if the log-minimal model conjecture is true, then the divisors contributing to the superpotential are always exceptional, in some sense (Section 6, Example 5.3 and Observation 1.3).

We study in detail the case of B Fano (i.e., $-c_1(B)$ is very ample): we give an explicit description of all the divisors contributing to the superpotential (Sections 4 and 1) and of the birational transformations of the Calabi–Yau 4-folds which contract these divisors. In Section 7 (Tables 1–7) we combine these (and other) results. What follows is a description of each section:

In Section 1 we describe properties of such divisors C which determine whether $D = \pi^*(C)$ contributes to the superpotential.

In Section 3 we describe our strategy for a systematic approach and develop an algorithm. The fundamental observation [9] is that these divisors cannot be *nef*, i.e., there is an effective curve Γ on B such that $C \cdot \Gamma < 0$. In particular, it follows from the results of Mori, Kollar and Kawamata that $C \cdot A < 0$, where A is the (homology) class of an effective curve on an "extremal ray" of the cone of effective curves of B (the dual of the Kähler cone). In Section 2 we define the cone of effective curves (the "*Mori cone*"), extremal rays, and properties which are relevant in our setup. These objects are, in fact, also the building blocks of the "Minimal model program" which, loosely speaking, is an algorithm to construct a "preferred" minimal model birationally equivalent to a given variety. It is exactly by following the steps of the algorithm that we can show that the divisors contributing to the superpotential are, in some sense, "exceptional" (which was hypothesized in [21,44]). We will return to this and the related birational transformations in Sections 5 and 6.

Our strategy consists in examining each extremal ray of the Mori cone and argue whether there exists an effective non-nef smooth divisor C such that $C \cdot \Gamma < 0$, with $\Gamma \in [R]$. This first step identifies all the possible non-nef divisors. Using the technical lemmas of Section 1 we can then determine the ones with the right numerical properties to contribute to the superpotential.

This gives a straightforward algorithm which can be applied any time the extremal rays of the effective cone of B are generated by effective curves; for example, when B is Fano, or B is toric, or a $\mathbb{P}^1$-bundle over certain surfaces. These cases are frequently considered as the basis of Calabi–Yau elliptic fibrations [21,28,29]. A byproduct of the above algorithm is a list of the fibrations $B \to S$, with general fiber isomorphic to $\mathbb{P}^1$ and S a surface. This is relevant from the point of view of the F-theory-heterotic duality.

Section 4 contains various examples. In particular we concentrate on an equidimensional elliptic Calabi–Yau $X \to B$, with B a Fano 3-fold ($c_1(B) > 0$) and show that the number of the divisors contributing to the superpotential is always finite. We use the classification of Mori–Mukai of such 3-folds to describe the divisors of type $D = \pi^*(C)$ which contribute to the superpotential for each Fano B, as well as the $\mathbb{P}^1$-fibrations (if any) of B (Section 7). We also compute the topoligical Euler characteristic of X, when $X = W$, its smooth Weierstrass model.

In Section 5 we will show how divisors contributing to the superpotential naturally are associated to faces of the Kähler cone of X which lead to another (necessarily singular) birational model of X.

Some of these divisors are indeed defined by birational contractions (to the Weierstrass model), see Observation 1.3. If $X = W \to B$ with B Fano, any $D = \pi^*(C)$ contributing to the superpotential determines a birational transformation $\phi : B \to B'$ with exceptional divisor C (i.e., $\operatorname{codim}(\phi(C)) \geq 2$). We construct a *flop* of X along D, and then contract the image of D. We obtain an elliptic Calabi–Yau 4-fold (over B'), with "canonical singularities" (the same type of singularities as of the Weierstrass model of a Calabi–Yau). I do not know whether it is possible to build a physical model with these singularities. It is possible in various examples (Section 5) to smooth the singularities and obtain another Calabi–Yau, where there is *no longer a contribution to the superpotential* related to D. In this case, the Hodge numbers change.

In Section 6 we speculate that this is always the case, even in M-theory. In fact, a generalized (but still conjectural) version of the minimal model algorithm implies that given any divisor D contributing to the superpotential on a Calabi–Yau 4-fold X (in M- or F-theory), there is a birational model of the fibration $\rho(X)$ such that $\rho(D)$ does not contribute to the superpotential and $\rho(X)$ is singular (at least with *canonical singularities*). This generalized version (the "log-minimal model program") has been worked out in various cases, among which are toric varieties.

The divisors contributing to the superpotential thus generate reflections of the Kähler cone of X in a "larger" cone. It would be interesting to study the Weyl group generated by such reflections, and how this is (if at all) related to the heterotic duals and the change of Hodge numbers of the smoothed variety (as in [33,34]).

It would also be interesting to study the Calabi–Yau 4-fold which can be "connected" by transitions related to divisors contributing to the superpotentials: see also [1,2,39].

The core of the paper is in Sections 3, 4 and 6.2: the reader should probably start with Section 3 and the general strategy, continue with the examples (Sections 4, 5 and 6.2 and the tables (Section 7) and use Sections 1.2 and 6.1 as a reference.

Finally, in writing this paper I had to give the precedence to some topics over others. The parts left out will be investigated in a sequel (in the not too distant future, I hope).

1. Technicalities

The motivation of this paper comes from describing the divisors contributing to the superpotential in F-theory; in this context our results are complete and more satisfactory at the moment. Nevertheless, many of the properties stated here apply also to M-theory.

We start by considering a smooth elliptic Calabi–Yau n-fold X with a section; i.e., $K_X = \mathcal{O}_X$, $h^i(\mathcal{O}_X) = 0$, $0 \leq 1 \leq n-1$, and there is a morphism $\pi : X \to B$ to a smooth

$(n-1)$-dimensional variety with general fiber a smooth elliptic curve. Furthermore, there exists a morphism $s : B \to T \subset X$ which is isomorphic to its image ("the section" of π), with inverse $\pi|_T$. (It is actually enough, for many of the applications considered here, to consider a "rational" section, i.e., the inverse s of π is only defined on an open set in B).

We also assume that the elliptic fiber degenerates over a non-trivial divisor in B. As a consequence, $h^i(B, \mathcal{O}_B) = 0\ \forall i > 0$; if $\dim(B) = 2$, B is rational; if $\dim(B) = 3$, B is uniruled.

Definition 1.1. $\pi_0 : W \to B$ is a *Weierstrass model* if W can be described by the homogenous equation $y^2 z = x^3 + Axz^2 + Bz^3$ in the projective bundle $\mathbb{P}(\mathcal{O} \oplus L^2 \oplus L^3)$, with L a line bundle on B and A and B sections of $-4L$ and $-6L$, respectively.

If $L = -K_B$, then $K_W \sim \mathcal{O}_W$.

W is often singular; interesting mathematics and physics arise from the resolutions of singularities, see for example [4,17,34]. On the other hand, if $-K_B$ is very ample and $h^i(B, \mathcal{O}_B) = 0\ \forall i > 0$ then W is a smooth Calabi–Yau manifold. Many elliptic Calabi–Yau manifolds can be constructed in this way (see Section 7).

If $\pi : X \to B$ is an elliptic Calabi–Yau with section, we can assume (without loss of generality) that $\pi : X \to B$ is the resolution of a Weierstrass model $\pi_0 : W \to B$. In fact:

Lemma 1.2. *Let $X \to B$ be a smooth, elliptic Calabi–Yau n-fold, with B smooth. Then:*

(1) *$K_X = \pi^*(K_B + \Delta)$, where $12\Delta = \sum n_i \Sigma_i$, $n_i \in \mathbb{N}$. Here Σ_i denotes a component of the locus in B where the elliptic curve degenerates; the summation is taken over all such components.*

(2) *If the fibration π has a section $B \to X$, then there exists a Weierstrass model of the fibration and a birational morphism μ such that $K_X = \mu^*(K_W)$ (i.e., W has "canonical" singularities) and the following diagram is commutative:*

$$\begin{tikzcd} X \arrow[rr, "\mu"] \arrow[dr, "\pi"'] & & W \arrow[dl, "\pi_0"] \\ & B. & \end{tikzcd}$$

(See also Section 6.1.)

Proof. (1) and existence of the Weierstrass model are due to Nakayama's Theorem 2.1 [37]; a discussion of the notation can be found in [34, vol. 473]. A straightforward argument shows that the condition $K_X \sim \mathcal{O}_x$ implies that $K_X = \mu^*(K_W)$. This proves (2). □

In F-theory a divisor D contributes to the superpotential only if it is "vertical", i.e., $\pi(D)$ is not the whole B [44]. In this paper we analyze the divisors $D = \pi^*(C)$, where C is a smooth curve on B. The motivation is given by the following:

Observation 1.3. *Let D be a smooth divisor in X, as above, contributing to the superpotential.*

(1) *If $\pi(D)$ is not a divisor on B, D is necessarily exceptional for μ.*
If $\pi : X \to B$ is equidimensional, then either:
(2) *$D = \pi^*(C)$, where $C \subset B$ is a smooth irreducible divisor such that $\pi_D : D \to C$ is an elliptic fibration, or*
(3) *$\pi(D) = \Sigma_i$ is a smooth component of the ramification divisor and either D is exceptional for μ or W is singular along a subset of $\mu(D)$.*

Proof. The existence of a section, equidimensionality and smoothness of D force $\pi(D)$ to be smooth. If $D = \pi^*(C)$, then a simple computation shows that the fiber over a general point of C is a smooth elliptic curve and thus $\pi_D : D \to C$ is an elliptic fibration in the sense of Kodaira (see also [34, vol. 473]; in fact it is enough to consider the Weierstrass model). □

Observation 1.4. *Conversely; if $D = \pi^*(C)$ is a smooth divisor, with C smooth, then $\pi_D : D \to C$ is an elliptic fibration.*

Note that the divisors of types (1) and (3) in Observation 1.3 are always *finite* in number and *exceptional*, in some sense. As this paper was being written [17] appeared, where a particular class of divisors of type in Observation 1.3 is studied. They show in particular that under certain hypothesis, some D are not exceptional for μ but contribute to the superpotential (as in (3) Observation 1.3); this is why we write "exceptional in some sense" (see also Section 6.1).

One application of this work is a criterion to determine under which conditions the divisors of type (2) in Observation 1.3 are also finite in number and exceptional. If π is not *equidimensional*, then $\pi(D)$ might not be smooth, when D is smooth (see Section 5.3 for an example). It might be that one should consider, more generally divisors with mild singularities (see also [11]). On the other hand, π is indeed equidimensional in many of the examples considered in F-theory.

We should also point out that $\chi(D)$ and $h^i(D, \mathcal{O}_D)$ are birational invariants and Nakayama [37] showed that there exists a smooth birationally equivalent elliptic fibration which is equidimensional over the strict transform of C. We plan to discuss this topic in a continuation of this paper.

In the rest of this section we study properties of the divisors D of type (3) in Observation 1.3.

Our goal is to reduce the calculation on B, by writing $\chi(D)$ as an expression on B. This is particularly useful when the geometry of B is well known, for example B is a toric or Fano variety.

(In the following $h^k(V, \mathcal{L}) = 0$, whenever $k < 0$.)

Lemma 1.5. *Let $X \to B$ be a smooth, elliptic Calabi–Yau n-fold, with B smooth and let $C \subset B$ and $D \subset X$ be smooth divisors such that $D = \pi^*(C)$. Then*

$$h^m(D, \mathcal{O}_D = \begin{cases} h^m(C, \mathcal{O}_C) + h^{n-1-m}(C, \mathcal{O}_C(C)) & \forall 0 \le m \le n-1, & (1.1) \\ h^m(C, \mathcal{O}_C) + h^{m-1}(C, -\Delta_{|C}) & \forall 0 \le m \le n-1, & (1.2) \end{cases}$$

$$\begin{aligned} \chi(D, \mathcal{O}_D) &= \chi(C, \mathcal{O}_C) + (-1)^{n-1}\chi(C, \mathcal{O}_C(C)) \\ &= \chi(C, \mathcal{O}_C) + (-1)^{n-1}\chi(C, K_C + \Delta). \qquad (1.3) \end{aligned}$$

Proof. Note that $\pi_{|D} = {}^{\mathrm{def}}\pi_D : D \to C$ is an elliptic fibration between smooth varieties; let us set $\Delta_C = \Delta_{|C}$. By Lemma 1.2, $12\Delta_C$ is a line bundle supported on the ramification locus of the fibration, which is the complement in B of the locus of the image of the smooth elliptic curves of the fibration.

Theorems 2.1, 7.6 and 7.7 in [23] apply to $\pi_D : D \to C$ and we get the short exact sequences:

$$0 \to H^k(C, (\pi_D)_*(K_D)) \to H^k(D, K_D) \to H^{k-1}(C, K_C) \to 0, \\ 0 \le k \le n-2,$$

which give

$$h^k(D, K_D) = h^k(C, \pi_D * (K_D)) + h^{k-1}(C, K_C), \quad \forall 0 \le k.$$

By the adjunction formula and Lemma 1.2 (1) the following equalities hold:

$$\begin{aligned} K_D &= (K_X + D)_{|D} = (\mathcal{O}_X + D)_{|D} = \pi^*(C)_{|D} \\ &= \pi_D{}^*(C_{|C}) = \pi_D{}^*(\mathcal{O}_C(C)) = \pi^*(K_B + \Delta + C)_{|D} \\ &= \pi_D{}^*((K_B + C)_{|C} + \Delta_C) = \pi_D{}^*(K_C + \Delta_C). \end{aligned}$$

The projection formula [13] now gives $\pi_{D*}(K_D) = K_C + \Delta_C = C_{|C}$. Note that $C_{|C} = N_{C/B}$ is the normal bundle of C in B.

The statement of the lemma follows from Serre's duality, applied to $V = C$ (resp. $V = D$) and $L = K_C + \Delta_C$, K_C (resp. $L = K_D$).

(*Serre's duality*: $h^m(V, L) = h^{r-m}(K_V - L)$, where L a line bundle on a smooth r-dimensional variety V.) □

Combining the results in Lemma 1.5 we also get the following corollary, which will be used in the explicit computations.

Corollary 1.6. *In the hypothesis of Lemma 1.5, assume that* $\dim(X) = 4$. *Then:*

$$
\begin{aligned}
h^0(D,\mathcal{O}_D) &= h^0(C,\mathcal{O}_C),\\
h^1(D,\mathcal{O}_D) &= h^1(C,\mathcal{O}_C)+h^2(C,C_{|C})\\
&= h^1(C,\mathcal{O}_C)+h^2(B,C)\\
&= h^1(C,\mathcal{O}_C)+h^2(C,K_C+\Delta_C)\\
&= h^1(C,\mathcal{O}_C)+h^0(C,-\Delta_{|C}),\\
h^2(D,\mathcal{O}_D) &= h^2(C,\mathcal{O}_C)+h^1(C,C_{|C})\\
&= h^2(C,\mathcal{O}_C)+h^1(B,C)\\
&= h^2(C,\mathcal{O}_C)+h^1(C,K_C+\Delta_C)\\
&= h^2(C,\mathcal{O}_C)+h^1(C,-\Delta_{|C}),\\
h^3(D,\mathcal{O}_D) &= h^0(C,C_{|C})=h^0(B,C)-1\\
&= h^2(C,K_C+\Delta_{|C})=h^2(C,-\Delta_{|C}).
\end{aligned}
\tag{1.4}
$$

Furthermore,

$$\chi(D,\mathcal{O}_D)=-1/2(K_C+\Delta_C)\cdot\Delta_C=1/2K_B\cdot C^2. \tag{1.5}$$

Proof. When X is a 4-fold, by the Hirzebruch–Riemann–Roch theorem for a line bundle L on a smooth surface C we have

$$\chi(C,L)=\chi(\mathcal{O}_c)+\tfrac{1}{2}(L-K_C)\cdot L$$

and obtain the first equality in (1.5) by substituting $L=K_C+\Delta_C$.

From the short exact sequences

$$
\begin{aligned}
&0\to\mathcal{O}_B(-C)\to\mathcal{O}_B\to\mathcal{O}_C\to 0,\\
&0\to\mathcal{O}_B\to\mathcal{O}_B(C)\to\mathcal{O}_C(C)\to 0,
\end{aligned}
$$

we find

$$\chi(D,\mathcal{O}_D)=2\chi(\mathcal{O}_B)-\chi(\mathcal{O}_B(-C))-\chi(\mathcal{O}_B(C)). \tag{1.5a}$$

On the other hand, the Hirzebruch–Riemann–Roch theorem for a line bundle L on a smooth 3-fold B says that

$$\chi(B,L)=\chi(\mathcal{O}_B)+\tfrac{1}{6}L^3-\tfrac{1}{4}L^2\cdot K_B+\tfrac{1}{12}L\cdot(K_B^2+c_2).$$

Substituting this expression in (1.5a) for $L=C$ and $L=-C$, respectively, we obtain the second equality.

The first set of equalities in (1.4) are a direct consequence of Lemma 1.5. The second set follows from (1.5a), since $h^i(B,\mathcal{O}_B)=0\ \forall i>0$. □

The following corollary is the first application of the above machinery; it is obvious, but useful in computation.

Corollary 1.7.
(1) *If* $\Delta_C = \mathcal{O}_C$, *then* D *does not contribute to the superpotential.*
(2) *If* $\Delta_C \neq \mathcal{O}_C$, *then* $h^1(D, \mathcal{O}_D) = h^1(C, \mathcal{O}_C)$.
(3) $h^3(D, \mathcal{O}_D) = 0 \Leftrightarrow h^0(B, C) = 1$.

Proof. (1) If $\Delta_C = \mathcal{O}_C$, then $\chi(D, \mathcal{O}_D) = 0$, by (1.5).

(2) Recall that 12Δ is an effective divisor. If $\Delta_C \neq \mathcal{O}_C$, $h^0(\Delta_C) \neq 0$ would imply $h^0(-12\Delta_C) \neq$ and thus $h^0(12\Delta_C) = 0$ (if a divisor and its opposite have non-zero sections, then the divisor is necessarily trivial). □

2. Minimal model theory and the superpotential

The extremal rays in the sense of Mori are relevant in this case; we fix some notation and recall some results of Mori (et al.). Standard references are [7,19,43]. B denotes any smooth complex algebraic variety. In Sections 3 and 4 we will apply the facts stated here to the case of B, the smooth base 3-fold of an elliptic Calabi–Yau 4-fold fibration (see also Section 6).

First we give some definitions.

Definition 2.1. By $\mathrm{NE}(B) \subset \mathbb{R}^l$ we denote the cone generated (over $\mathbb{R}_{\geq 0}$) by the effective cycles of (complex) dimension 1, mod. numerical equivalence; and by $\overline{\mathrm{NE}(B)}$ its closure in the finite-dimensional real vector space $\mathbb{R}^\ell$ of all cycles of complex dimension 1, mod. numerical equivalence (see for example [7]).

Note that $\ell = rk(\mathrm{Pic}(B))$, and in the cases we are considering here $\ell = b_2(B)$, the second Betti number of B.

Kleiman's criterion [20] says that D is *ample* if and only if $D \cdot \Gamma > 0$ for all $\Gamma \in \overline{\mathrm{NE}(B)}$; in particular $\overline{\mathrm{NE}(B)}$ is the dual of the closure of the ample cone with the duality given by the intersection pairing between curves and divisors.

The closure of the ample cone is called the *nef* cone: a divisor D is *nef* if and only if $D \cdot \Gamma \geq 0$ for all the effective curves Γ on B.

The description of $\mathrm{NE}(B)$ for many varieties B can be found in [7,19,26], and for Fano 3-folds (the case considered in Example/Theorem 4.5, Corollaries 4.6 and 4.7 and Example 4.8) in [27]. We present here some examples that will be relevant in Section 4, in the description of divisors contributing to the superpotential:

Example 2.2. If $rk(\mathrm{Pic}(B)) = 1$, then $\overline{\mathrm{NE}(B)}$ is the positive real half-line.

Example 2.3. If $B = \mathbb{F}_n$ is a ruled rational surface $B \to \mathbb{P}^1$, then $\overline{\mathrm{NE}(B)}$ is the convex cone generated by $\{f, \sigma_\infty\}$, where f is the (class of the) fiber of the fibration and by σ_∞ the (class of the) unique section with $\sigma_\infty^2 = -n$.

Example 2.4. If $B = \mathbb{P}^1 \times S$ and $\overline{\mathrm{NE}(S)}$ is generated by $\{f_i\}$, then $\overline{\mathrm{NE}(B)}$ is generated by $\{\ell, f_i \times t\}$, where ℓ is the class of the fiber of the projection $B \to S$ (which is a smooth $\mathbb{P}^1$) and t is a point in $\mathbb{P}^1$.

Definition 2.5. R is called a *negative extremal ray* on the smooth 3-fold B if R is an extremal ray of the cone $\overline{\mathrm{NE}(B)}$ in the usual sense, and $K_B \cdot A < 0$ for a curve (equivalently, for all curves) A with homology class spanning the extremal ray R. We will write $A \in [R]$.

Example/Theorem 2.6 (Mori). *If $-K_B$ is ample, then every extremal ray is negative and $\overline{NE(B)}$ is the convex cone generated by the extremal rays. Furthermore, the extremal rays are finite in number.*

Theorem 2.7 (Contraction Theorem, see for example [7, 43]). *If R is a negative extremal ray, then there exists a morphism $\phi_R : B \to B_R$, where B_R is a projective variety with "mild" singularities (which can be described), and an irreducible curve $E \subset B$ is contracted by ϕ_R if and only if the homology class of E belongs to the external ray R. Furthermore, $rk(\mathrm{Pic}(B)) > rk(\mathrm{Pic}(B_R))$; ϕ_R is called the contraction morphism.*

(The singularities which occur are called "*terminal*"; if $\dim(B) = 2$ these are the smooth points.)

Remark 2.8. In general, if any morphism contract a curve on one extremal ray, then it necessarily contracts all the effective curves on the same extremal ray.

Example 2.9 (Contraction morphisms and extremal rays). In Example 2.2, if $K_B \cdot \Gamma < 0$, for an effective curve Γ on B, (equivalently, all effective curves) then $\mathrm{NE}(B)$ consists of one negative extremal ray and the corresponding contraction morphism sends B to a point. If $K_B \cdot \Gamma \geq 0$, then there is no negative extremal ray.

In Example 2.3, f is always a negative extremal ray ($K_B \cdot f = -2$) and the corresponding contraction morphism gives the structure of $\mathbb{P}^1$-bundles $\mathbb{F}_n \to \mathbb{P}^1$.

On the other hand, $K_B \cdot \sigma_\infty < 0$ only when $n = 1$; in this case σ_∞ in the only negative extemal ray. The corresponding contraction morphism is $\mathbb{F}_n \to \mathbb{P}^2$ the blow up of $\mathbb{P}^2$ at a point. Note that we can always contract σ_∞, independently of the value of n. The image surface however will always be singular unless $n = 1$. In fact the "mild" singularities mentioned above (in the statement of the contraction theorem) are exactly the smooth points when $\dim(B) = 2$.

Example/Theorem 2.10 (Mori). *If $\dim B = 3$, then the exceptional locus C_R of a birational morphism $\phi_R : B \to B_R$ associated to a negative extremal ray R is one of the following reduced divisors:*

(1) *C_R is a $\mathbb{P}^1$-bundle over the smooth curve $\phi_R(C_R)$, with $\frac{1}{2} K_B \cdot C_R^2 = 1 - g(\phi_R(C_R))$;*
(2) *$C_R \sim \mathbb{P}^2$ with $\mathcal{O}_{C_R}(C_R) = \mathcal{O}_{\mathbb{P}^2}(-1)$;*
(3) *$C_R \sim \mathbb{P}^2$ with $\mathcal{O}_{C_R}(C_R) = \mathcal{O}_{\mathbb{P}^2}(-2)$;*
(4) *$C_R \sim \mathbb{P}^1 \times \mathbb{P}^1$ with $\mathcal{O}_{C_R}(C_R) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$;*
(5) *C_R is a singular quadric surface in $\mathbb{P}^3$.*

In cases (1) and (2) B_R is a non-singular 3-fold; $\phi_R(C_R)$ is a quadruple point in case (3) and a double point otherwise.

Proof. See [7,30], or [43]. □

3. The algorithm

We now consider the case of $\pi : X \to B$, an elliptic fibration of a Calabi–Yau 4-fold X, with C and $D = \pi^*(C)$ smooth divisors. This happens when π is an equidimensional elliptic fibration with section, as we saw in Observation 1.3.

The following remarks are the building blocks of our strategy:

Remark 3.1. If $D = \pi^*(C)$ contributes to the superpotential, then there exists an extremal ray R on $\overline{\mathrm{NE}(B)}$ such that $C \cdot A < 0$ for all the curves A on the ray R.

In fact, D and C cannot be nef divisors [9]; C is non-nef if and only if $C \cdot A < 0$ for all A on some extremal ray R of $\overline{\mathrm{NE}(B)}$.

Remark 3.2. If C is not *nef*, all the curves on the extremal ray R must be contained in C. In particular, *if* there exists a morphism $B \to S$ contracting exactly the curves on the extremal ray R, then $\dim(B) = \dim(S)$.

Strategy.
(S1) We consider cases where the extremal rays of $\overline{\mathrm{NE}(B)}$ are generated by effective curves.
(S2) For each extremal ray R, we determine whether there exists an effective smooth divisor C such that $C \cdot \Gamma < 0$, for Γ on the ray R.
(S3) If such a C exists, we check its numerical properties.

In most relevant cases (in F-theory) this strategy gives a quick algorithm to determine the divisors of this form contributing to the superpotential. We will do so explicitly in Section 4.

In fact, the extemal rays generate the cone of effective curve when B is Fano (cf. Definition 2.5), toric [3,38], or a $\mathbb{P}^1$-bundle over certain surfaces. These cases are frequently considered as the basis of Calabi–Yau elliptic fibrations (cf. (S1)).

Often the extremal rays are defined in terms of morphisms (see Remark 2.8); this is in fact always the case for negative extremal rays (by the contraction theorem) (cf. (S2)).

At the same time (by looking at the extremal rays) we can also describe the $\mathbb{P}^1$-bundle structure (if any) of B. This is relevant from the point of view of daulity with heterotic theory. We will do so explicitly in Sections 4 and 7 (Tables 1–7).

We will use Corollaries 1.6 and 1.7 for (S3).

Another advantage of this approach is that in our examples we get a map

$$\{\text{divisors contributing to the superpotential}\}$$
$$\to \{\text{faces of the Kähler cone of } X\}.$$

Note that the divisors of types (1) and (3) in Observation 1.3 are always associated to a face of the Kähler cone of X as they come from resolving the Weierstrass model of X.

For the divisors of type (2) in Observation 1.3 the question is more subtle:

Remark 3.3. In terms of the dual "*nef*" cone, the morphism associated to a chosen negative extremal ray gives a divisor class on the boundary of the "*nef* cone" of B (cf. Theorem 2.7). When we start from a divisor contributing to the superpotential, this "face" of the nef cone must lead to another (birational) model of B by Remark 3.2.

In Section 5 we will show how divisors contributing to the superpotential are associated to faces of the Kähler cone of X which lead to another (necessarily singular) birational model (of X).

We speculate that this is always the case, even in M-theory and show how this is related to various conjectures in algebraic geometry [19].

3.1. The case of negative extremal rays

Only the negative extemal rays which determine birational contractions (Remark 3.2) are relevant for our purposes.

In this case ($\dim(B) \leq 3$ there is a unique non-nef divisor C_R such that $C_R \cdot \Gamma < 0$ $\forall \Gamma \in [R]$ (cf. Example/Theorem 2.10); we also have a complete list of the possible C_R which occur. We only consider here smooth divisors C_R (see [44]); the case (5) of the Example/Theorem 2.10, the quadric cone in $\mathbb{P}^3$ should be also of interest, as it is a divisor of simple normal crossings [11]. This will be investigated in a forthcoming paper.

The following proposition follows directly from (Corollary 1.6) together with Mori's description given in Example/Theorem 2.10.

Proposition 3.4. *Let R be a negative extremal ray associated to a birational morphism ϕ_R, C_R the unique exceptional divisor and $D_R = \pi^*(C_R)$, as in Example/Theorem* 2.10. *In cases* (2)–(4)

$$h^0(D_R, \mathcal{O}_{D_R}) = 1,$$
$$h^1(D_R, \mathcal{O}_{D_R}) = h^2(D_R, \mathcal{O}_{D_R}) = h^3(D_R, \mathcal{O}_{D_R}) = 0,$$

and D_R always contributes to the superpotential.

In case (1),

$$\chi(D_R) = \tfrac{1}{2} K_B \cdot C_R^2 = 1 - g(\phi_R(C_R)) = 1$$

if and only if $\phi_R(C_R)$ is a rational curve; furthermore,

$$\begin{aligned} h^0(D_R, \mathcal{O}_{D_R}) &= 1, \qquad h^1(D_R, \mathcal{O}_{D_R}) = h^1(C, \mathcal{O}_C) = 0, \\ h^2(C, \mathcal{O}_{D_R}) &= \chi(D_R, \mathcal{O}_{D_R}) - 1 - h^1(C, \mathcal{O}_C) \\ &= h^3(D_R, \mathcal{O}_{D_R}) = 0, \end{aligned}$$

and D_R contributes to the superpotential if and only if C_R is rationally ruled.

We speculate that these are exactly the cases when C_R deforms whenever B deforms.

4. Examples (the algorithm at work)

We apply the algorithm outlined in Section 3 to various examples. because we restrict ourselves to divisors of type (2) of Observation 1.3 (i.e., of the form $D = \pi^*(C)$, with D and C both smooth), we describe on B the relevant divisors C (such that $D = \pi^*(C)$ contributes to the superpotential). If $X = W$, then these are all the divisors contributing to the superpotential; and we can write the superpotential (Section 7). The divisors not of this form are always finite in number (in F-theory), are "exceptional in some sense (see Observation 1.3), and can be described with other methods.

Example 4.1. If $b_2(B) = rk(\mathrm{Pic}(B)) = 1$, no divisor of the form $D = \pi^*(C)$ contributes to the superpotential and there is no fibration $B \to S$, $\dim(S) \neq 0$.

In fact, in this case, NE(B) is a half-line (see Example 2.2): any divisor containing all of the line would also contain all of B.

In particular, $-K_B = c_1(B)$ is necessarily an ample divisor: in fact $-12K_B$ is the effective, non-trivial, divisor image (under π) of the singular fibers. B is a Fano 3-fold; such varieties were classified by Iskovskih [14,15]. Among those are $B = \mathbb{P}^3$ and $B = Q$ the smooth quadric in $\mathbb{P}^4$. The complete list will appear in Section 7.

Example 4.2 (no. 27, Table 3). If $B = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, no divisor of the form $D = \pi^*(C)$ contributes to the superpotential.

In fact, $\overline{\mathrm{NE}(B)}$ is a cone with three edges in $\mathbb{R}^3$: each edge being a fiber of the projection to two of the factors (see Example 2.4 and Remark 3.2).

Example 4.3 (for $n = 1$, no. 28, Table 3). If $B = \mathbb{P}^1 \times \mathbb{F}_n$, $n \geq 1$, then no divisor contributes to the superpotential when $n \neq 1$, and one divisor contributes when $n = 1$. In the latter case, the divisor is determined by a negative extremal ray of type (1) of Example/ Theorem 2.10.

There is a $\mathbb{P}^1$-fibration $B \to \mathbb{F}_n$ and a $\mathbb{P}^1$-fibration $B \to \mathbb{P}^1 \times \mathbb{P}^1$.

(This is analyzed in [44], for $n = 1$.)

In fact, NE(B) is generated by $\{\ell, f, \sigma\}$ where ℓ is a fiber of $p: B \to \mathbb{F}_n$, f a fiber of $B \to \mathbb{P}^1 \times \mathbb{P}^1$, and $\sigma = \sigma_\infty \times \{t\}$, $t \in \mathbb{P}^1$, as in Examples 2.3 and 2.4.

There is no non-nef divisor associated to f or ℓ cf. Remark 3.1; if we set $C = p^{-1}(\sigma_\infty)$, then $C \cdot \sigma = -n$ is non-nef when $n > 0$. Note that $C \sim \mathbb{P}^1 \times \mathbb{P}^1$ and $h^0(C, \mathcal{O}_C) = 0$, $h^1(C, \mathcal{O}_C) = h^2(C, \mathcal{O}_C) = 0$; we identify the Picard group of $\mathbb{P}^1 \times \mathbb{P}^1$ with (a, b): a curve is effective when $a, b \geq 0$.

We need to compute $h^i(C, C_{|C})$ (for $n > 0$) and determine whether C contributes to the superpotential, by Corollary 1.6. A simple computation gives $C_{|C} = (-n, 0)$ and immediately $h^0(C, C_{|C}) = 0$, $h^2(C, C_{|C}) = h^0(C, (n-2, -2)) = 0$ (by Serre's duality). It follows also that $h^1(C, C_{|C}) > 0$ if $n \geq 2$ and $h^1(C, C_{|C}) = 0$ for $n = 1$ (Kunneth's formula).

Example 4.4 [9]. If $B = S \times \mathbb{P}^1$, where S is a general rational elliptic surface with section, then there are infinitely many divisors contributing to the superpotential, corresponding to the negative extremal rays of $NE(S)$. There is one $\mathbb{P}^1$-fibration to $\mathbb{P}^1 \times \mathbb{P}^1$ and one to S.

It is not hard to see that the generators for $\overline{\mathrm{NE}(S)}$ are f and s_α, where f is a fiber and s_α a section of the elliptic fibration $p : S \to \mathbb{P}^1$. We can choose S so that the s_α's are infinitely many; it turns out that every s_α is a negative extremal ray.

Then $NE(B)$ (see Example 2.4) is generated by $\{f \times t, s_\alpha \times t, \ell\}$ with s_α, f as above, $t \in \mathbb{P}^1$, and l a fiber of $B \to S$.

$f \times t$ and ℓ do not determine divisors contributing to the superpotential: they are in fact the general fibers of $B \to \mathbb{P}^1 \times \mathbb{P}^1$ and $B \to S$, respectively (cf. Remark 3.2), while $s_\alpha \times t$ is a negative extremal ray on B for all α. The corresponding divisor C_{S_α} is of type (1) of Example/Theorem 2.10, is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and contributes to the superpotential. It follows that the divisors contributing to the superpotential are exactly the C_{s_α} (cf. Remark 3.1).

Example/Theorem 4.5. *Assume that B is a Fano variety, i.e. $c_1(B) = -K_B$ is ample. Then the divisors $D = \pi^*(C)$ contributing to the superpotential are exactly the exceptional divisors $C = C_R$ (corresponding to the contraction of extremal ray R) listed below:*

(1) *C_R is a $\mathbb{P}^1$-bundle over a smooth rational curve $(\phi_R(C_R))$,*

(2)–(3) *$C_R \sim \mathbb{P}^2$,*

(4) *$C_R \sim \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. If B is Fano, the negative extremal rays generate $\overline{\mathrm{NE}(B)}$; thus C is non-nef only it contains all the curves E in the homology classes of some negative extremal ray R. Mori's result says that all such curves span the exceptional locus of the morphism ϕ_R, described in Example/Theorem 2.10. Then necessarily, $C = C_R$ and ϕ_R is a divisorial contraction. The statement follows from Proposition 3.4. □

Corollary 4.6. *Let B be a Fano 3-fold.*

(1) *If π is equidimensional, then there is only a finite number of divisors contributing to the superpotential.*

(2) *If $X = W$ (the Weierstrass model is smooth), then the divisors contributing to superpotential are exactly the exceptional divisors of Mori contractions listed above.*

Proof. If B is Fano (by the "Cone theorem"), there are only finitely many negative extremal rays, hence a finite number of such divisors D on X contributing to the superpotential. □

In the examples in [44] Witten shows that "a superpotential is not generated by instantons by showing that any divisor D on X has $\chi(D) \neq 1$, or we show that a superpotential is generated by showing that some choice of the cohomology class there is precisely one complex divisor D, which moreover has $h_1 = h_2 = h_3 = 0$".

This is exactly what always happens for B Fano:

Corollary 4.7. *Let B be a Fano 3-fold and $D = \pi^*(C)$. Then either $\chi(D) \neq 1$ or $h^0(D) = 1$, $h^i(D) = 0$, $i \neq 0$.*

Mori–Mukai [31] classified all Fano 3-folds and Matsuki [27] described the extremal rays for each of them: the relevant divisors are the one corresponding to birational contractions (cf. Remark 3.2). We apply our algorithm to each case in their list and we determine the divisors of type (2) of Observation 1.3 contributing the superpotential (Section 7). The only delicate point is when C is of type (1) Example/Theorem 2.10, i.e., a $\mathbb{P}^1$-bundle over a smooth curve $L = \phi_R(C) : C$ contributes if and only if L is rational (cf. Proposition 3.4). The following identity is useful to compute $g(L)$ the genus of L (notation as in Example/Theorem 2.10):

$$(-K_B)^3 = (-K_{B_R})^3 - 2\{-K_{B_R} \cdot L - g(L) + 1\}.$$

Example 4.8 ($b_2 = 3$, no. 9 in [31]). B is the blow up of the cone over the Veronese surface $R_4 \subset \mathbb{P}^5$ with center a disjoint union of the vertex and the quartic in $R_4 \sim \mathbb{P}^2$. (Recall that the Veronese surface is $\mathbb{P}^2$ embedded in $\mathbb{P}^5$ by its linear system of conics.)

The Matsuki–Mori–Mukai classification says that NE(B) is generated by 4 curves (the extremal rays): R_1, the ruling of the exceptional divisor over the quartic, R_2, the strict transform of a line in the Veronese surface; R_3, the ruling of the exceptional divisor which is the strict transform of the ruling over the quartic, R_4, a line in the exceptional divisor of the blow up of the vertex of the Veronese cone. Furthermore, the corresponding extremal contractions $\phi_{R_i} : B \to B_{R_i}$ are all birational: R_1 and R_3 are of type (1) of Example/Theorem 2.10, while R_2 and R_4 are of type (3) of Example/Theorem 2.10. The exceptional divisors of ϕ_{R_i}, $i = 2, 4$, contribute to the superpotential, while the others are $\mathbb{P}^1$-bundles over a curve of positive genus (the plane quartic) and do not contribute (cf. Example/Theorem 4.5).

B_{R_i}, $i = 1, 3$, is the blow up of the Veronese cone with center a plane quartic; while B_{R_i}, $i = 2, 4$, is isomorphic to the blow up of the Veronese cone with center the vertex ($b_2 = 2$, no. 36). The extremal transition with exceptional divisors contributing to the superpotential lead in this case to a *singular* variety B_R.

5. Transitions of CY 4-folds I: the case of negative extremal rays

One of the examples studied by Witten [44] is B the blown up of $\mathbb{P}^3$ at a point (this is no. 35 in the list of Mori–Mukai). A puzzle arises here: while there is no divisor on $\mathbb{P}^3$ contributing to the superpotential, the exceptional divisor of the blow up contributes on B.

We will show that all the divisors contributing to the superpotential are always "exceptional" in some sense, at least when B is Fano. The general statement depends on the (log)-minimal model conjecture which will be discussed in Section 6.

Note that some of these divisors are actually "defined" by birational contractions (to the Weierstrass model), see Observation 1.3. The divisors considered in [17] are of this form.

What will follow is in fact a four-dimensional analog of the construction in [34, vol. 476]: in that case $B = \mathbb{F}_1$, C_R is the curve with self-intersection -1 (this curve is in fact a negative extremal ray, cf. Example 2.2). Morrison and Vafa perform a toric flop of the holomorphic image of C_R in X ($X \to B$ has a section), and then contract the image of $D = \pi^{-1}(C)$ (which is a Del Pezzo surface). Finally, they smooth the singularity.

Similarly, we consider $\pi : X \to B$, with X equal to its smooth Weierstrass model, B Fano and assume that $D = \pi^*(C)$ is a divisor contributing to the superpotential. Then $C = C_R$ is the exceptional divisor of the contraction morphism $\phi_R : B \to B_R$ associated to the negative extremal ray R (Section 3 and Example/Theorem 4.5, Corollaries 4.6 and 4.7 and Example 4.8). D cannot be contracted immediately (see Example 6.8), so (as in [34, vol. 476]) we first start with a "flop" (Proposition 5.1) and follow with a contraction (Proposition 5.2).

As in [34] we assume, for simplicity's sake, that the fibration π is general, i.e. that there is only one section.

The new 3-fold $\overline{X}_R \to B_R$ is elliptically fibred, but it is singular: in the cases where X is a general smooth Weierstrass model (as in [34, vol. 476]), the singularities can be described precisely. These type of singularities are called "canonical" (Section 6) and are the same type of singularities that occur on the *singular* Weierstrass models. It is not clear to me whether a physical model can be built with these singularities.

If the singularity can be smoothed (we explicitly do so in various cases), then the resulting Calabi–Yau will have different Hodge numbers.

It is an interesting question to investigate this change and how it might be related to the exceptional divisor contracted (as in [34, vol. 476]).

Proposition 5.1 (The flop). *Let $C = C_R$ as in Example/Theorem 4.5 and Corollary 4.6. Then there exists a contraction $B \to B_R$, where B_R is another 3-fold and a birational transformation ("flop") $X \to X_R$ such that the following diagram is commutative:*

$$\begin{tikzcd} X \arrow[r, "\mu_R"] \arrow[d, "\pi"'] & X_R \arrow[d, "\pi_R"] \\ B \arrow[r, "\phi_R"] & B_R \end{tikzcd}$$

X_R is smooth only if C_R is of type (1) *in Example/Theorem* 2.10.

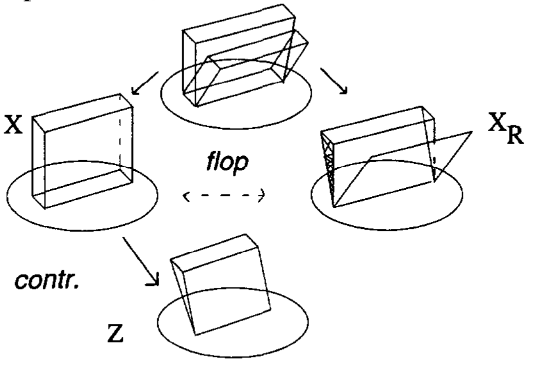
Proof (Following Matsuki). We have assumed the existence of a section of the elliptic fibration; so there exists a smooth 3-fold T isomorphic to B in X (a "copy" of B in X); by C_R we will denote both the surface in B and its isomorphic image in T. We can "duplicate" the contraction of C_R in B cf. Example/Theorem 2.10 in its holomorphic image in T (and X) and obtain a birational transformation $X \to Z$, see Example 6.8). Matsuki in [27] considers a similar situation and explicitly constructs the flop of each surface C_R in X, for each R. The pictures are fairly self-explanatory:

- the large ovals denote T, the image of B in X and its images after the blow ups and blow downs,
- the object in the ovals denote the image of C_R in X and their images after the various birational transformations,
- the "parachute type" objects in the X and X_R denote D and its image D_R after the "flop".

It is clear from the picture that D_R has intersection positive with the fiber of the contraction with Z, while $D \cdot R < 0$ (R is the fiber of the contraction $X \to Z$). We have performed a "*log-flip*" with respect to D (see also Example 6.8).

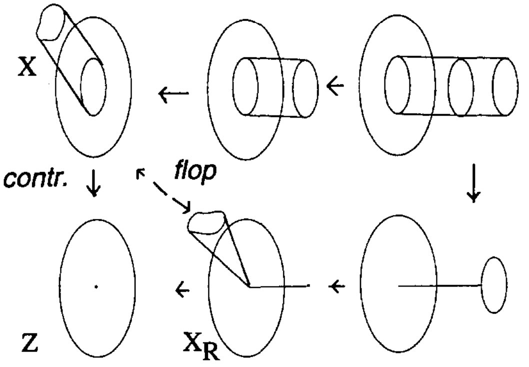
For a detailed description see [27, pp. 30–36] and also Section 6.

C_R is of type (1) in Example/Theorem 2.10: C_R is $\mathbb{P}^1$-bundle over $\phi_R(C)$, and B_R is a smooth 3-fold. The shaded area is a vertical "section" of $\mu(D) = D_R$, which is isomorphic to the Del Pezzo surface which is obtained by blowing up $\mathbb{P}^2$ at eight points (see also Proposition 5.2, Example 6.8 and [34, vol. 476]).



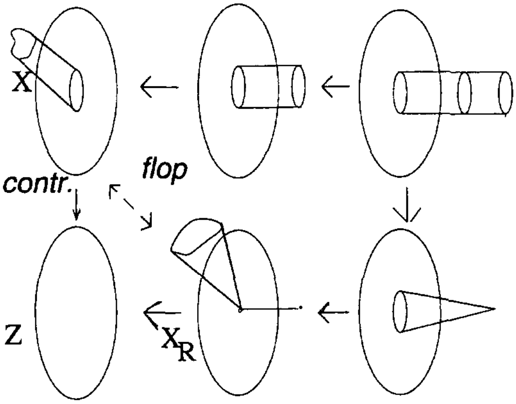
X_R is smooth.

C_R is of type (in Example/Theorem 2.10): $C_R \sim \mathbb{P}^1 \times \mathbb{P}^1$, $\phi_R(C_R)$ is an ordinary double point in B_R.



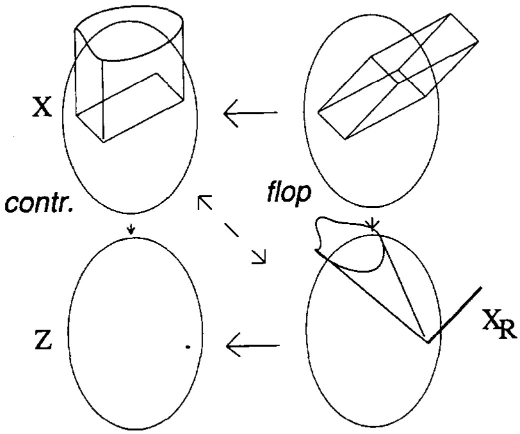
X_R is singular along a $\mathbb{P}^1$ (the "fat" point in the picture).

C_R is of type (3) in Example/Theorem 2.10: $C_R \sim \mathbb{P}^2$, $\phi_R(C_R)$ is a quadruple point in B_R.



X_R has a singular point (the "fat" point in the picture).

C_R is of type (4) in Example/Theorem 2.10: $C_R \sim \mathbb{P}^2$, B_R is non-singular.



X_R has a singular point (the "fat" line in the picture). □

As Matsuki points out these are not the only flops which can occur; however our goal here is to show that we can ultimately contract the image of D, which cannot be contracted in X (see Section 6). However, it is possible that one would need to consider other type of flops to describe all the reflections (and corresponding Weyl group) of the Kähler cone of X, in the enlarged Kähler cone, determined by the divisor contributing the superpotential.

Note the flops used above are toric, even when B is not toric.

Proposition 5.2 (The contraction). *There is birational transformation $\rho : X \to \bar{X}_R$ with exceptional divisor $\rho(D)$ and an elliptic fibration $\bar{X}_R \to B_R$ (with section) such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\rho} & \bar{X}_R \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}_R} \\ B & \xrightarrow{\phi_R} & B_R. \end{array}$$

$\bar{X}_R$ has canonical singularities ($\rho^*(K_{\bar{X}_R}) = K_X$).

If the Weierstrass model of X is singular, it has canonical singularities; I do not know if one can construct a physical model with these singularities.

If $\bar{X}_R$ can be smoothed, then the Hodge numbers of the resulting manifolds will be different.

Proof of Proposition 5.2. We describe in details the case (1) of Example/Theorem 2.10; the others are similar. See also Section 6. C_R is a $\mathbb{P}^1$-bundle over the rational curve $\phi_R(C_R)$, with fiber f while $\mu_R(C_R)$ is a surface. The elliptic fibration $\pi_S: \pi^{-1}(f) = S \to f$ is a rational elliptic surface with section (see [33,34]), for each fiber f; the section is given by the intersection of C_R with S. After the "flop" S is a Del Pezzo surface $\mu_R(S)$, isomorphic to the blown up of $\mathbb{P}^2$ at eight points. Each surface can be contracted to a point; actually all the surfaces can be simultaneously contracted to a rational curve Γ_R (see Example 6.8), with a birational morphism $X_R \to \bar{X}_R$. Let $\rho: X \to \bar{X}_R$ denote the compositions of the two birational morphisms; from the explicit construction of the flop it is clear that the elliptic fibration over B_R is preserved and the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\rho} & \bar{X}_R \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\ B & \xrightarrow{\phi_R} & \bar{B}_R. \end{array}$$

Note that $\operatorname{codim} \rho(D) \geq 2$, i.e., the image of the divisors contributing to the superpotential is no longer a divisor. On the other hand $\bar{X}_R$ is singular along Γ_R; these singularities are *canonical* (like the singularities of the Weierstrass model of X). [K,1.5]. X_R is equisingular along Γ_R: the singularity at each point of Γ_R of a transverse 3-fold is exactly as in vol. 476 of [34]. In fact, we can smooth $\bar{X}_R$ as in [34, vol. 476].

The transitions among Fano 3-folds with exceptional divisors contributing to the superpotential appear in Section 7.

Example 5.3 (*Where have all these divisors gone?*). From the tables in Section 7, we can see the sort of the other divisors contributing to the superpotential after a birational contraction Propositions 5.1 and 5.2: some still contribute to the superpotential; in some other cases the birational morphism ϕ_R becomes a $\mathbb{P}^1$ (or conic bundle) fibration of B_R.

In Example 4.4. ($B = S \times \mathbb{P}^1$, with S a rational elliptic surface) the birational transformations ϕ_{R_α} corresponding to the extremal ray $s_\alpha \times t$ contracts B to $B_R = \mathbb{P}^1 \times S_1$, which is the unique Fano 3-fold with $b_2 = 10$ (S_k is the Del Pezzo surface obtained by blowing up $\mathbb{P}^2$ at $9 - k$ points; set $S = S_0$).

We can perform $0 \leq k \leq 9$ contractions of non-intersecting extremal rays and consider the induced elliptic fibration $\pi_k : X \to B_k = S_{9-k} \times \mathbb{P}^1$. If $k > 1$, there are D_k smooth divisors mapping to curves in B_k and π_K has no section. There is an infinite number of divisors contributing to the superpotential, whose image in B_k is a singular divisor. I do not know at this moment if this can occur when there is a section.

We can also contract the divisors D_k, as in Propositions 5.1 and 5.2. A finite number of such divisors will still contribute to the superpotential, while infinitely many become singular divisors with normal crossings.

6. Transitions of CY 4-folds II: are these divisors "exceptional"?

We present some evidence that all the divisors contributing to the superpotential (also in M-theory) are "exceptional", in the sense that are related to some birational transformation. They might not all be exceptional, in a strict sense, as one can see in the example considered in [17]. They show that under certain hypothesis, if $S \subset B$ is a rational surface and the "general" fiber over a point in S is a cycle of N rational curves with enhanced gauge group $SU(N)$, then each of the N irreducible component of $SU(N)$ contribute to the superpotential. However, only $N - 1$ of them are "exceptional" divisors. In this case the birational morphism is the contraction to the Weierstrass model (Observation 1.3). It should be pointed out that there exists a relation among these N divisors ($N - 1$ are "independent") [17].

If the normal bundle is negative (Grauert), a contraction is possible, at least in the analytic category. We would like this contraction to be projective and to describe the singularities which might occur. In the case of F-theory we would also like to preserve the elliptic structure.

Our approach is to consider the pair (X, D), where D is a divisor contributing to the superpotential and exploit once more the fact that this divisor cannot be *nef* (Remark 3.1). We will need some more general definition given in Section 2.

The reader should probably start from the second part of this section ("the general case") and use the first one as a reference.

6.1. *Log-minimal models*

There are several versions of the log-minimal model program; we follow [19], as it seems at the moment to be the best suited for our applications.

$\pi : X \to B$ is any proper morphism between varieties; later we will apply the general machinery to the case of the elliptic Calabi– Yau.

Definition 6.1. $\overline{\mathrm{NE}(X/B)} \subset \mathbb{R}^m$ is the closed convex cone generated (over $\mathbb{R}_{\geq 0}$) by the effective cycles of (complex) dimension 1, mod. numerical equivalence.

Definition 6.2. D is π-nef if $D \cdot \Gamma \geq 0$ for all the curves $\Gamma \in \overline{\mathrm{NE}(X/B)}$.

A *relative* version of Kleiman's criterion says that the cone of π-*nef* divisors (which is the closure of the π-ample cone) and $\overline{\mathrm{NE}(X/B)}$ are dual cones. The duality is again given by the intersection pairing (cf. Definition 2.1).

Is what follows we will have to consider singular varieties; a crucial point in the (log)-minimal model program is the existence of a "reasonable" intersection pairing between complex curves (with values in $\mathbb{Q}$) and complex subvarieties of codimension 1 (*Weyl divisors*). This motivates the following:

Definition 6.3. A variety has $\mathbb{Q}$-factorial singularities if for any D Weil divisor, there exists an integer r such that rD is a line bundle (D is also called $\mathbb{Q}$-Cartier divisor).

Unless noted otherwise all the varieties are assumed to be normal and $\mathbb{Q}$-factorial. We will also consider Weyl divisors with rational coefficients.

Below $\mathcal{D}$ is such a divisor: $\mathcal{D} = \sum a_i L_i$, with L_i distinct complex subvarieties of codimension 1 (Weyl divisors) and $a_i \in \mathbb{Q}, 0 \leq a_i < 1$. $\bigcup L_i$ is called *support* of $\mathcal{D}$.

We write $\mathcal{D} \equiv \mathcal{D}'$ if some multiple of $\mathcal{D}$ and $\mathcal{D}'$ are equivalent as line bundles.

Definition 6.4. The pair $(X, \mathcal{D})$ (as above) has at worse *log-terminal* (*log-canonical*) singularities if there exists a resolution of the singularities $f : Y \to X$ such that the union of the exceptional divisor and the inverse image of $\cup L_i$ is a divisor with normal crossings and

$$K_Y \equiv f^*(K_X + \mathcal{D}) + \sum b_k M_k$$
$$\text{such that } b_k > -1 \text{ (resp. } \geq -1), \quad \forall k.$$

(The definition does not depend on the choice of f and Y.) If $\mathcal{D} = 0$ and $b_k > 0 (b_k \geq 0)$ then the singularities are called *terminal* and *canonical*, respectively.

If $\dim(X) = 2$ and X, the singularities are at worse terminal, then X is smooth, the canonical singularities are the rational double points.

The following is a generalized version of the contraction theorem (Theorem 2.7):

Theorem 6.5 (Contraction morphism). *Let $\pi : X \to B$ be a morphism between varieties. If $(X, \mathcal{D})$ has log-terminal singularities and $K_X + \mathcal{D}$ is not π-nef (i.e. $(K_X + \mathcal{D}) \cdot R < 0$ for some extremal ray $R \in \overline{\mathrm{NE}(X/B)}$), then there exists a morphism that $\psi_R : X \to Z$, contracting all the curves in the numerically equivalence (homology) class of $[R]$ such that the following diagram is commutative:*

$$\begin{tikzcd} X \arrow[rr, "\psi_R"] \arrow[dr, "\pi"'] & & Z \arrow[dl, "\pi_+"] \\ & B & \end{tikzcd}.$$

Z is a normal variety and $\dim \overline{\mathrm{NE}(X/B)} > \dim \overline{\mathrm{NE}(Z/B)}$.

Proof. For a proof and various reference, see for example [19, Theorems 3.1.1, 3.2.1, 4.1.1 and 4.2.1]. □

(∗) We assume also that some line bundle multiple of $K_X + \mathcal{D}$ has a section (i.e., the Kodaira dimension of $K_X + \mathcal{D}$ is non-negative). This is the case in our applications, where $K_X \sim \mathcal{O}_X$ and a multiple of $\mathcal{D}$ is an effective divisor contributing to the superpotential.

In this case, the contraction morphism in Theorem 6.5 is birational.

The *log-minimal model conjecture* says that there exists a birational map $\rho : X \to \bar{X}$ and a morphism $\bar{\pi} : \bar{X} \to B$ such that $K_{\bar{X}} + \bar{\mathcal{D}}$ is $\bar{\pi}$-nef and the following diagram is commutative:

$$\begin{tikzcd} X \arrow[rr, "\rho"] \arrow[dr, "\pi"'] & & \bar{X} \arrow[dl, "\bar{\pi}"] \\ & B & \end{tikzcd} .$$

Here $\bar{\mathcal{D}} = \rho(\mathcal{D})$ and $(\bar{X}, \bar{\mathcal{D}})$ is the *log-minimal model*.

The problem is that when the contraction in Theorem 6.5 is not divisorial (i.e. the exceptional locus is not a divisor), it is not possible to define an intersection product which is compatible with our structure (Definition 6.3). If so, we would in fact have a contradiction:

$$0 > (K_X + \mathcal{D}) \cdot R = \psi_R^*(K_Y + \bar{\mathcal{D}}) \cdot R = (K_Z + \bar{\mathcal{D}}) \cdot \psi_R(\mathcal{D}) = 0.$$

In this case we have the following:

Conjecture 6.6. *There is another birational transformation ("log-flip"), which is an isomorphism outside a set of codimension greater than* 2 *(an isomorphism in codimension* 1*):*

$$\begin{tikzcd} \mathbf{X} \arrow[rr, dashed, leftrightarrow, "\text{log-flip}"] \arrow[dr] \arrow[ddr, bend right, "\pi"'] & & \mathbf{X_R} \arrow[dl] \arrow[ddl, bend left, "\pi_R"] \\ & \mathbf{Z} & \\ & \mathbf{B} & \end{tikzcd}$$

such that X_R has log-terminal ($\mathbb{Q}$ – factorial) singularities and $\mu(K_{X_R} + \mu(\mathcal{D})) \cdot R_+ > 0$, for all the curves A contracted by ψ_+.

The number consecutive to such log-flips is always finite.

The log-minimal model conjecture is a theorem if $\dim(X) \leq 3$ (see for example [25]) and it has been worked out in various special examples, among which the ones considered in Section 5 (which we review in Example 6.8) and when the techniques of toric geometry can be applied [27,38].

6.2. *Transition II: The general case*

Now let X be a smooth Calabi–Yau 4-fold and D a divisor contributing to the superpotential; then D is not nef, i.e., there is an effective (complex) curve R such that $D \cdot R = (K_X + D) \cdot R < 0$ (cf. Remark 3.1). The idea is to use the contraction morphism in Theorem 6.5.

We consider the pair $(X, \mathcal{D})$, where $\mathcal{D} = rD$, for some $0 < r < 1, \in \mathbb{Q}$: X is smooth, so we can take as f in Definition 6.4 the identity map and verify that the pair has log-terminal singularities (this is true also if D has normal crossing singularities).

If the log-minimal model conjecture holds, then the following conjecture is true:

Conjecture 6.7. *Let X be a Calabi–Yau 4-fold and D a divisor contributing to the superpotential. Then there exists a birational transformation $\rho : X \dashrightarrow \bar{X}$, with canonical singularities (the same singularities of the Weierstrass model) and $\rho(D)$ is a nef effective divisor.*

Proof. Start with $(X, \mathcal{D})$ as above. If the log-minimal model conjecture holds, then ρ is a composition of contraction morphisms (Theorem 6.5) and log-flips (Conjecture 6.6). If $\nu : X \dashrightarrow X'$ is either the contraction morphism in Theorem 6.5, or the "log-flip" in Conjecture 6.6 then X' has canonical singularities [18, 1.5]; these are the same singularities of our Weierstrass models (cf. Lemma 1.2)). Then $K'_x \sim \mathcal{O}_{X'}$ and $K_{X'} + \nu(\mathcal{D}) \sim \nu(\mathcal{D})$.

Note that $\nu(\mathcal{D})$ is well-defined and that these log-flips are "flops" (because the canonical divisor is trivial). □

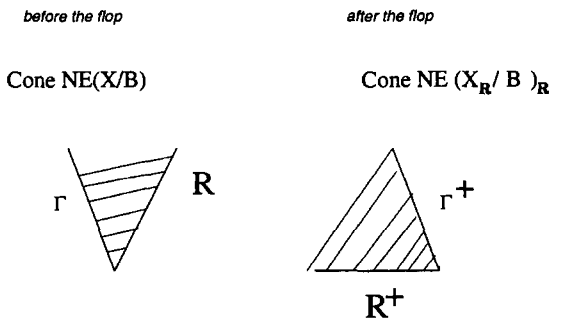
Propositions 5.1 and 5.2 are particular cases of this general setup.

Example 6.8. Let $\pi : X \to B$ an elliptic fibration between smooth varieties. Assume that X is equal to the smooth "general" Weierstrass model over B and that D is divisor contributing to the superpotential. Then $D = \pi^*(C)$ for some smooth divisor on B.

Now let us consider the induced elliptic fibration $\epsilon : X \to B_R$ and $\overline{\mathrm{NE}(X/B_R)}$. This two-dimensional cone is generated by a fiber Γ of the fibration π and the extremal ray R in X (more precisely, the isomorphic image in the section $T \subset X$ of the extremal ray R in B): $D \cdot \Gamma = 0$, while $D \cdot \Gamma < 0$.

Then there exists a contraction morphism $\psi_R : X \to Z$ (Theorem 6.5) contracting the curves in the homology of class $[R]$; this contraction cannot be divisorial (it comes from a contraction from the lower dimensional B). In each of the cases considered the flop $\mu_R : X \dashrightarrow X_R$ exists [27]. Let π_R be the induced elliptic fibration. We now concentrate on the case (1) of Example/Theorem 2.10; the others are similar.

After the "flop" the relative cone $\overline{\mathrm{NE}(X_R/B_R)}$ is still two-dimensional and it is generated by the image of the fiber of π_R, which we will denote by Γ^+, and R_+, a fiber of $X_R \to Z$. It is easy to verify that $\mu(D) \cdot R_+ > 0$, while $\mu(D) \cdot \Gamma^+ < 0$. In this case the contraction morphism corresponding to Γ^+ is divisorial and the divisor $\mathcal{D}$ is the exceptional divisor.



If $\pi : X \to B$ has more than one section (the rank of the Mordell–Weyl group is positive), we first have to perform "flops" along the sections (as in [34, vol. 476]).

7. Tables for "general" elliptic CY with basis Fano 3-folds

In the following $\pi : X \to B$ is an elliptic Calabi–Yau 4-fold and B is a Fano 3-fold. In Tables 1–6 we follow the list of Fano 3-folds of Iskovskih–Mori–Mukai: the 3-folds are subdivided by their second betti number, $1 \leq b_2 = h^{1,1} \leq 10$, which is also the dimension of the Mori (and Kähler) cone of B.

We gather various information about the Fano 3-folds and the "general" elliptic Calabi–Yau 4-folds fibred over them.

We use the criteria developed in Sections 1 and 3 to determine the divisors of the form $D = \pi^*(C)$ which contribute to the superpotential on X (cf. Example 4.8). If $X = W$, the smooth Weierstrass model (Definition 1.1), these divisors are all the divisors contributing to the superpotential (cf. Observation 1.3). The divisors C determine a birational contraction $B \to B_R$. We identify B_R when it is another Fano (cf. Example/Theorem 4.5). If $X = W$, we also compute the topological Euler characteristic of X. In particular:

- The first number in the table corresponds to the one assigned in [31] to each 3-fold with a given $b_2 = h^{1,1}$. If $\pi : W = X \to B$ is the smooth general Weierstrass model (Definition 1.1) (there is only one section of π), then $h^{1,1}(X) = h^{1,1}(B) + 1$.
- The second column says whether B is toric: a list of toric 3-fold and the related superpotential appears in [21,29], if the 3-fold is toric, $\mathcal{F}_k$ is the symbol used in [3,29,38]; many examples are also in [28].
- The third column is about the divisors contributing to the superpotential as in Example/Theorem 4.5. If $D_j = \pi^*(C)$, then (C) is a divisor of the type (j), $1 \leq j \leq 4$, if there are two different divisors of the same type (j) of Example/Theorem 2.10 we will denote them as D_j^1, D_j^2 $(1 \leq j \leq 4)$.

  If the same divisors are exceptional for two different contractions (as in Example/Theorem 4.5), we simply write it twice (this is the case of no. 3, $h^{1,1} = 3$)

- If $\phi : B \to B_R$ is the contraction of D_j then $h^{1,1}(B_R) = h^{1,1}(B) - 1$. If $X = W$ these birational transformations are "promoted" to birational transitions of the Calabi–Yau 4-fold X (see Sections 5 and 6.2).
  If B_R is Fano, $D_j(l)$ means that B_R is the Fano 3-fold with number ℓ in the Mori–Mukai classification [31] of 3-folds with $h^{1,1} = h^{1,1}(B) - 1$.
- The fourth column lists the $\mathbb{P}^1$-fibrations (denoted by $p_i : B \to S$) and the conic bundles (denoted by $c_i : B \to S$): this is relevant from the point of view of heterotic theory.
- The fifth column is $12c_1(B) \cdot c_2(B) + 360c_1^3(B)$, which is the Euler characteristic of the smooth Weierstrass model (if any) over B. By the Riemann–Roch theorem for 3-folds [13], $12c_1(B) \cdot c_2(B) = 288_\chi(\mathcal{O}_B) = 288$ (B is uniruled).
- We use a rather crude (but readily available [14,15,32]) criterion to determine whether there exists a smooth Weierstrass Calabi–Yau model over B (cf. Definition 1.1), namely we require $-K_B$ to be very ample. On the other hand, most Fano satisfy this criterion: we write "no" in the last column if $-K_B$ is not very ample. Otherwise $X = W$, its smooth Weierstrass model; in this case, we see from the list that

$$\chi(X) = 144(17 + 5\ell), \quad 0 \le \ell \le 25, \quad \ell = 28, 29.$$

- Table 7 is the flow chart of transition among the Fano 3-folds corresponding to divisors contributing to the superpotential (as in Example/Theorem 4.5). These are also "promoted" to transition among Calabi–Yau 3-folds (as in Sections 5 and 6.2).
  The columns correspond the values of $h^{1,1}(B)$, starting from 5 on the left and ending with 1 on the right.
  The thick lines represent a contraction of a divisor ($\sim \mathbb{P}^1 \times \mathbb{P}^1$) to a point (cf. type (2) of Example/Theorem 2.10), while the others represent a contraction of a rational ruled surface (cf. type (1) of Example/Theorem 2.10).
  Table 1 represents $h^{1,1}(B) = 1$.
  Tables 2 and 3 contain data for $h^{1,1}(B) = 2$ and 3, respectively.

Table 1
$h^{1,1}(B) = 1$

Iskovskih [14,15] classified all such varieties: the following occur in the flow chart (Table 7), together with $\mathbb{P}^3$:
- $Q \subset \mathbb{P}^4$, a smooth quadric surface.
- $V_3 \subset \mathbb{P}^3$, a smooth cubic surface.
- $V_4 \subset \mathbb{P}^5$, a complete intersection of two quadrics.
- $V_5 \subset \mathbb{P}^9$, is a complete intersection of a linear subspace $\mathbb{P}^6 \subset \mathbb{P}^9$ and the Grassmann variety of Gr 1, 4 embedded in $\mathbb{P}^9$ by the Plucker embedding.
  - The only Fano toric variety with $h^{1,1}(B) = 1$ is $\mathbb{P}^3$.
  - No divisor contribute to the superpotential (cf. Example 4.1) and there is no $\mathbb{P}^1$-fibration.
  - All these 3-folds have $-K_B$ very ample with the following exceptions (see [15, vol. 12, Table 6.5] or also [36]):
- The double cover of $\mathbb{P}^3$ with branch locus a sextic.
- The double cover of a quadric in $\mathbb{P}^4$ branched over the intersection of the quadric and a quartic.
- V_1 (i.e. the double cover of the cone over the Veronese).
- V_2 (i.e. the double cover of $\mathbb{P}^3$ with quartic ramification).

Table 2
$h^{1,1}(B) = 2$

No.	Toric	Contribution to the superpotential	Fibrations	$\chi(X)$	Very ample
1	No	None	None		No
2	No	None	$c : B \to \mathbb{P}^2$		No
3	No	None	None		No
4	No	None	None	3888	
5	No	None	None	4608	
6	No	None	$c_1 : B \to \mathbb{P}^2, c_2 : B \to \mathbb{P}^2$	4608	
7	No	None	None	5328	
8a	No	D_4	$c : B \to \mathbb{P}^2$	5328	
8b	No	None	$c : B \to \mathbb{P}^2$	5328	
9	No	None	$c : B \to \mathbb{P}^2$	6048	
10	No	None	None	6048	
11	No	D_1 (V_3)	$c : B \to \mathbb{P}^2$	6768	
12	No	None	None	7488	
13	No	None	$c : B \to \mathbb{P}^2$	7488	
14	No	None	None	7488	
15a	No	D_4	None	8208	
15b	No	None	None	8208	
16	No	D_1 (V_4)	$c : B \to \mathbb{P}^2$	8208	
17	No	None	None	8928	
18	No	None	$c : B \to \mathbb{P}^2$	8928	
19	No	D_1 (V_4)	None	9648	
20	No	D_1 (V_5)	$c : B \to \mathbb{P}^2$	9648	
21	No	D_1^1 (Q), D_1^2	None	10 368	
22	No	D_1^1 $(\mathbb{P}^3)$, D_1^2 (V_5)	None	11 088	
23a	No	D_4	None	11 088	
23b	No	None	None	11 088	
24	No	None	$c : B \to \mathbb{P}^2, p : B \to \mathbb{P}^2$	11 088	
25	No	None	None	11 808	
26	No	D_1^1 (Q), D_1^2 (V_5)	None	12 528	
27	No	D_1 $(\mathbb{P}^3)$	$p : B \to \mathbb{P}^2$	13 968	
28	No	D_3	None	14 688	
29	No	D_1 (Q)	None	14 688	
30	No	D_1 $(\mathbb{P}^3)$, D_2	None	16 848	
31	No	D_1 (Q)	$p : B \to \mathbb{P}^2$	16 848	
32	No	None	$p_1 : B \to \mathbb{P}^2,\ p_2 : B \to \mathbb{P}^2$	17 568	
33	$\mathcal{F}_5$	D_1 $(\mathbb{P}^3)$	None	19 728	
34	$\mathcal{F}_2$	None	$p : B \to \mathbb{P}^2$	19 728	
35	$\mathcal{F}_3$	D_2 $(\mathbb{P}^3)$	$p : B \to \mathbb{P}^2$	20 448	
36	$\mathcal{F}_4$	None	$p : B \to \mathbb{P}^2$	22 608	

S_k denotes $\mathbb{P}^2$ blown up at $9 - k$ points; for example $\mathcal{F}_{13} = \mathbb{P}^1 \times S_7$. Tables 4 and 5 contain data for $h^{1,1}(B) = 4$ and 5, respectiely.

$B = \mathbb{P}^1 \times S_k$, with $1 \leq k \leq 5$; $h^{1,1}(B) = 11 - k$. None of these 3-folds is toric; the extremal contractions are induced by the blow ups: $S_k \to S_{k+1}$. Table 6 shows data when $h^{1,1}(B) \geq 6$.

Table 3
$h^{1,1}(B) = 3$

No.	Toric	Contribution to the superpotential	Fibrations	$\chi(X)$
1	No	None	$c_1 : \mathbb{P}^1 \times \mathbb{P}^1, c_2 : \mathbb{P}^1 \times \mathbb{P}^1, c_3 : \mathbb{P}^1 \times \mathbb{P}^1$	4608
2	No	D_1, D_1	$c : B \to \mathbb{P}^1 \times \mathbb{P}^1$	5328
3	No	None	$c : B \to \mathbb{P}^1 \times \mathbb{P}^1$	6768
4	No	D_1 (18)	$c_1 : B \to \mathbb{P}^2 . c_2 : B \to \mathbb{P}^1 \times \mathbb{P}^1$	6768
5	No	D_1^1 (34), D_1^2, D_1^2	None	7488
6	No	D_1 (33)	$c : B \to \mathbb{P}^1 \times \mathbb{P}^1$	8202
7	No	None	None	8928
8	No	D_1^1 (24), D_1^2 (34)	$c : B \to \mathbb{P}^1 \times \mathbb{P}^1$	8928
9	No	D_3^1, D_3^2	None	9648
10	No	D_1^1 (29), D_1^2	None	9648
11	No	D_1^1 (25), D_1^2 (34)	None	10 368
12	No	D_1^1 (27), D_1^2 (33), D_1^3(34)	None	10 368
13	No	D_1^1 (32), D_1^2, D_1^3	None	11 088
14	No	D_2(28), D_3	None	11 808
15	No	D_1^1 (29), D_1^2 (31), D_1^3(34)	None	11 808
16	No	D_1^1 (27), D_1^2 (32), D_1^3(35)	None	12 528
17	No	D_1^1 (34), D_1^2	$p : B \to \mathbb{P}^1 \times \mathbb{P}^1$	13 248
18	No	D_1^1(29), D_1^2 (30), D_1^3(33)	None	13 248
19	No	D_1^1 (35), D_1^2, D_1^3, D_1^4	None	13 968
20	No	D_1^1 (31), D_1^2 (32), D_1^3	None	13 968
21	No	D_1^1 (34), D_1^2, D_1^3	None	13 968
22	No	D_1^1 (34), D_1^2(36), D_3	None	14 688
23	No	D_1^1 (30), D_1^2 (31), D_1^3(35)	None	15 408
24	No	D_1^1 (32), D_1^2 (34)	$p : B \to \mathbb{F}_1$	15 408
25	$\mathcal{F}_8$	D_1^1 (33), D_1^2	$p : B \to \mathbb{P}^1 \times \mathbb{P}^1$	16 128
26	$\mathcal{F}_{12}$	D_1^1 (33)D_1^2(34), D_2(35)	None	16 848
27	$\mathcal{F}_6$	None	$p_1 : \mathbb{P}^1 \times \mathbb{P}^1, p_2 : \mathbb{P}^1 \times \mathbb{P}^1, p_3 : \mathbb{P}^1 \times \mathbb{P}^1$	17 568
28	$\mathcal{F}_9$	D_1 (34)	$p_1 : B \to \mathbb{P}^1 \times \mathbb{P}^1, p_2 : B \to \mathbb{P}^1 \times \mathbb{P}^1$	17 568
29	$\mathcal{F}_{11}$	D_1^1 (35), D_1^2 (36), D_3	None	18 288
30	$\mathcal{F}_{10}$	D_1^1 (33), D_1^2(35)	$p : B \to \mathbb{F}_1$	18 288
31	$\mathcal{F}_7$	D_1, D_1	$p : B \to \mathbb{P}^1 \times \mathbb{P}^1$	19 008

Table 4
$h^{1,1}(B) = 4$

No.	Toric	Contribution to the superpotential	Fibrations	$\chi(X)$
1	No	None	None	8928
2	No	$D_1^1, D_1^1, D_1^2, D_1^2$	None	10 368
3	No	$D_1^1(17), D_1^2(27), D_1^3(28), D_1^4(28)$	None	11 088
4	No	$D_1^1(18), D_1^2(18), D_1^3(19), D_1^4(30), D_1^5(30)$	None	11 808
5	No	$D_1^1(21), D_1^2(28), D_1^3(31), D_1^4, D_1^5$	None	11 808
6	No	$D_1^1(25), D_1^2(25), D_1^3(25), D_1^4(27)$	None	12 528
7	No	$D_1^1(24), D_1^2(24), D_1^3(28), D_1^4(28)$	None	13 248
8	No	$D_1^1(27), D_1^2(31), D_1^3(31), D_1^4(31)$	None	13 968
9	$\mathcal{F}_{15}$	$D_1^1(25), D_1^2(26), D_1^3(28), D_1^4(30)$	None	14 688
10	$\mathcal{F}_{13}$	$D_1^1(27), D_1^2(28), D_1^3(28)$	$p: B \to S_7$	15 408
11	$\mathcal{F}_{14}$	$D_1^1(28), D_1^2(31), D_1^3, D_1^4$	None	16 128
12	$\mathcal{F}_{16}$	$D_1^1(30), D_1^2(30), D_1^3, D_1^4$	None	16 848

Table 5
$h^{1,1}(B) = 5$

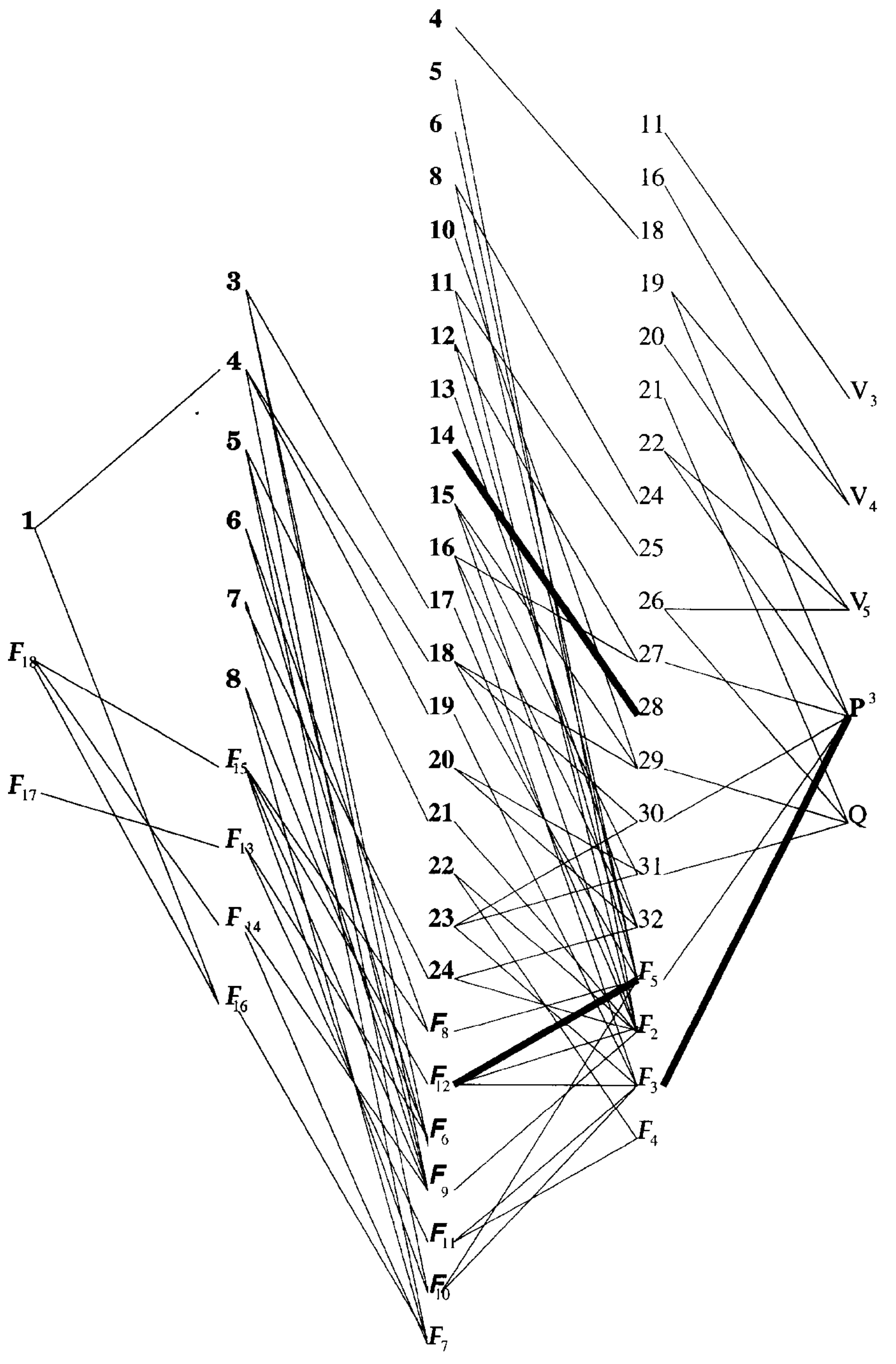
No.	Toric	Contribution to the superpotential	Fibrations	$\chi(X)$
1	No	$D_1^i(4), i = 1, 2, 3, D_1^i(12), i = 4, 5, 6, D_1^7$	None	10 368
2	$\mathcal{F}_{18}$	$D_1^i(9), i = 1, 2, D_1^i(11), i = 3, 4, D_1^5(12), D_1^i, i = 6, 7$	None	13 248
3	$\mathcal{F}_{17}$	$D_1^i(10), i = 1, \ldots, 6$	$p: B \to S_6$	13 248

Table 6
$h^{1,1}(B) \geq 6$

$h^{1,1}(B)$	Contribution to the superpotential	Fibrations	$\chi(X)$	Very ample
6	$D_1^i, i = 1, \ldots, 10$	$p: B \to S_5$	13 248	
7	$D_1^i, i = 1, \ldots, 16$	$p: B \to S_4$	15 408	
8	$D_1^i, i = 1, \ldots, 27$	$p: B \to S_3$	17 568	
9	$D_1^i, i = 1, \ldots, 56$	$p: B \to S_2$		No
10	$D_1^i, i = 1, \ldots, 240$	$p: B \to S_1$		No

Table 7 is the flow chart of transition among the Fano 3-folds corresponding to divisors contributing to the superpotential.

Table 7


1
F18
F17
3
4
5
6
7
8
F15
F13
F14
F16
4
5
6
8
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
F8
F12
F6
F9
F11
F10
F7
11
16
18
19
20
21
22
24
25
26
27
28
29
30
31
32
F5
F2
F3
F4
V3
V4
V5
P3
Q

Acknowledgements

I have discussed various parts of this project with R. Donagi, S. Katz and in particular D. Morrison and E. Witten. It is a pleasure to thank them all. Thanks also to P. Candelas, M. Larsen, A. Ksir and M. Schneier for all sorts of useful suggestions.

References

[1] A.C. Avram, P. Candelas, D. Jancic, M. Mandelberg, On the connectedness of the moduli space of Calabi–Yau manifolds, Nucl. Phys. 465 (1996) (hepth 9511230).
[2] A.C. Avram, M. Kreuzer, M. Mandelberg, H. Sharkey, The web of Calabi–Yau hypersurfaces in toric varieties (1997) (hepth 9703003).
[3] V.V. Batvrev, Toroidal Fano 3-folds, Izv. Akad. Nauk. 45 (1981) [Enlg. translation: Math. U.S.S.R., Izv. 19, 1982].
[4] M. Bershadsky, K. Intriligator, S. Kachru, D. Morrison, V. Sadov, C. Vafa, Geometric singularities and enhanced Gauges symmetries, Nucl. Phys. B 48 (1996) 215–252 (hepth 9605200).
[5] I. Brunner, M. Lynker, R. Schimmrigk, Unification of M- and F-theory Calabi–Yau fourfold vacua (1996) (hepth 9610195).
[6] I. Brunner, M. Lynker, R. Schimmrigk, Dualities and phase transitions for Calabi–Yau threefolds and fourfolds (1996) (hepth 9703182).
[7] H. Clemens, J. Kollár, S. Mori, Higher dimensional geometry, Asterisque 166 (1988).
[8] G. Curio, D. Lüst, A class of $N = 1$ dual string pairs and its modular superpotential (1997) (hepth 9703007).
[9] R. Donagi, A. Grassi, E. Witten, A non-perturbative superpotential with E_8 symmetry, Modem. Phys. Lett. A 17 (1996) (hepth 9607091).
[10] S. Forste, J. Louis, Duality in string theory (1996) (hepth 9612192).
[11] O. Ganor, A test of the chiral E_8 current algebra on a 6D non critical string, Nucl. Phys. B 479 (1996) (hepth 9607029).
[12] O. Ganor, A note on zeroes of superpotentials in F-theory, (1996) (hepth 9612077).
[13] R. Hartshorne, Algebraic Geometry, Springer, Berlin, 1977.
[14] V.A. Itskovskih, Fano 3-folds I, Izv. Akad. Nauk. 41 (1977) [English translation Math. U.S.S.R., Izv 11].
[15] V.A. Itskovskih, Fano 3-folds II, Izv. Akad. Nauk. 42 (1977) [English translation Math. U.S.S.R., Izv 12].
[16] S. Kachru, E. Silverstein, Singularities gauge dynamics, and nonperturbative superpotentials in string theory (1996) (hepth 9608194).
[17] S. Katz, C. Vafa, Geometric engineering of $N = 1$ quantum field theories (1996) (hepth 9611090).
[18] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degeneration of surfaces, Ann. of Math. 127 (1988) 93–163.
[19] Y. Kawamata, K. Matsuda, K. Matsuki, Introduction to the minimal model problem, Proc. Sym. Alg. Geom., Sendai, 1985, Adv. Stud. Pure Math., vol. 10, Kinokuniya, Tokyo, 1985, pp. 283–360.
[20] S. Kleiman, Toward a numerical theory of ampleness, Ann. of Math. 84 (1966) 293–344.
[21] A. Klemm, B. Lian, S.-S. Ruan, S.-T. Yau, Calabi–Yau fourfolds for M- and F- theory compactifications (1997) (hepth 9701023).
[22] K. Kodaira, Collected Works, vol. III, Princeton University Press, Princeton, NJ, 1975.
[23] J. Kollár, Higher dimensional images of dualizing sheaves, Ann. of Math. 123 (1986) 11–42.
[24] J. Kollár, Y. Miyaoka, S. Mori, Rationally connected varieties, J. Alg. Geom. 1 (1992) 429–428.
[25] J. Kollár et al., Log Flips and Abundance for Algebraic Threefolds, Astérisque 211 (1992).
[26] Y. Manin, Cubic forms: Algebra, Geometry, Arithmetic, North-Holland, Amsterdam, 1974.
[27] K. Matsuki, Weyl groups and birational transformations among minimal models, Memoirs of AMS 557 (1995).
[28] P. Mayr, Mirror symmetry, $N - 1$ superpotentials and tensionless strings on Calabi–Yau four-folds (1996) (hepth 9610162).

[29] K. Mohri, F-theory vacua in four dimensions and toric threefolds, (1997) (hepth 9701147).
[30] S. Mori, Threefolds whose canonical bundle are not numerically effective, Ann. of Math. 116 133–176.
[31] S. Mori, S. Mukai, Classifications of Fano 3-folds with $B_2 \geq 2$, Man. Math. 36 (1981) 147–162.
[32] S. Mori, S. Mukai, On Fano 3-folds with $B_2 \geq 2$. Adv. Stud. in Pure Math. 1 (1983) 101–129.
[33] D. Morrison, N. Seiberg, Extremal transitions and five-dimensional supersymmetric field theories, Phys. Lett. B 388 (1996) (hepth 9608111).
[34] D. Morrison, C. Vafa, Compactifications of F-theory on Calabi–Yau threefolds, I, II, Nucl. Phys. 473, 476 (1996) (hepth 9602114/9603161).
[35] S. Mukai, Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, Inv. Math. 77 (1984) 101–116.
[36] J.P. Murre, Classification of Fano threefolds according to Fano and Iskovskih, in: A. Conte (Ed.), Algebraic Threefolds Lecture Notes in Mathematics, vol. 947, Springer, Berlin, 1982, pp. 35–92.
[37] N. Nakayama, On Weierstrass models, in: Algebraic Geometry and Commutative Algebra, in honor of M. Nagata, Kinokunyia, 1987, pp. 405–431.
[38] T. Oda, Convex Bodies and Algebraic Geometry: An Introduction to the Theory of Toric Varieties, vol. 15, Springer, Berlin, 1988.
[39] M. Reid, The moduli space of 3-folds with $K = 0$ may nevertheless be irreducible, Math. Ann. 278 (1987) 329–334.
[40] J. Schwarz, Lectures on superstring and M-theory dualities (hepth 9607201).
[41] C. Vafa, Evidence for F-Theory Nucl. Phys. B 469 (1996) (hepth 9602022).
[42] C. Vafa, Lectures on strings and dualities Nucl. Phys. B 469 (1996) (hepth 9702201).
[43] P.M.H. Wilson, Towards birational classification of algebraic varieties, Bull. London Math. Soc. 19 (1987) 1–48.
[44] E. Witten, Non-perturbative superpotential in string theory, Nucl. Phys. B 474 (1996) (hepth 9604030).
[45] E. Witten, Five-brane effective action in M-theory (1996) (hepth 9610234).